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A q-ANALOGUE OF THE WIGNER–ECKART THEOREM FOR THE NONSTANDARD q-DEFORMED ALGEBRA $U'_q(\mathrm{so}_n)$

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Abstract

The tensor product of vector and arbitrary representations of the nonstandard q-deformation $U'_q(\mathrm{so}_n)$ of the universal enveloping algebra $U(\mathrm{so}_n)$ of Lie algebra so_n is defined. The Clebsch–Gordan coefficients of tensor product of vector and arbitrary classical or nonclassical type representations of q-algebra $U'_q(\mathrm{so}_n)$ are found in an explicit form. The Wigner–Eckart theorem for vector operators is proved.

1. Introduction

For the last fifteen years, much attention of mathematicians and mathematical physicists is attracted to the subject of quantum algebras and quantum groups. Besides the standard deformation of Lie algebras proposed by Drinfeld [1] and Jimbo [2], other (nonstandard) deformations are also under consideration. This paper deals with the deformation $U'_q(\mathrm{so}_n)$ of universal enveloping algebra $U(\mathrm{so}_n)$ proposed by Gavrilik and Klimyk [3]. Let us mention that the algebra $U'_q(\mathrm{so}_3)$ appeared earlier in the paper [4].

As a matter of interest, the algebras $U'_q(\mathrm{so}_n)$ arose naturally as auxiliary algebras in deriving the algebra of observables in 2+1 quantum gravity with 2D space of genus g, so that n depends on g, $n = 2g + 2$ [5,6,7].

As shown in [8], the algebra $U'_q(\mathrm{so}_n)$ admits a q-analogue of Gel'fand–Tsetlin formalism for construction of finite-dimensional irreducible representations. Since the algebra $U'_q(\mathrm{so}_n)$ is not a Hopf algebra, there is no a natural way to introduce the notion of tensor product of representations. But, as shown in [9,10,11], the algebra $U'_q(\mathrm{so}_n)$ is a subalgebra in Drinfeld–Jimbo Hopf algebra $U_q(\mathrm{sl}_n)$. Moreover, it is possible to show that the algebra $U'_q(\mathrm{so}_n)$ is a $U_q(\mathrm{sl}_n)$-comodule algebra such that the coaction coincides with the comultiplication in $U_q(\mathrm{sl}_n)$ if one embeds $U'_q(\mathrm{so}_n)$ into $U_q(\mathrm{sl}_n)$. This comodule structure can be used to introduce the tensor product of vector and arbitrary representations T of $U'_q(\mathrm{so}_n)$ (it will be denoted by $T^{\otimes}$), see [12].

We describe the decomposition of $T^{\otimes}$ into irreducible subrepresentations and write down the corresponding Clebsch–Gordan coefficients in the case when T is irreducible finite-dimensional representation of the classical or nonclassical type. The decomposition of $T^{\otimes}$ in the case of classical type representations has the same form as in the case of Lie algebra so_n and the corresponding Clebsch–Gordan coefficients are q-deformation of their classical analogues [13,14].

It is well-known that Wigner–Eckart theorem for the tensor operators with respect

to Lie algebra so_n (and, especially, so_3) is very important in physics. In this paper, we give a q–analogue of such theorem for the case of vector operators.

Everywhere below we suppose that q is not a root of unity.

2. The q-deformed algebra $U'_q(\mathrm{so}_n)$ and quantum algebra $U_q(\mathrm{sl}_n)$

According to [3], the nonstandard q-deformation $U'_q(\mathrm{so}_n)$ of the Lie algebra so_n is given as a complex associative algebra with $n-1$ generating elements $I_{21}, I_{32}, \ldots, I_{n,n-1}$ obeying the defining relations

$$\begin{aligned} &I^2_{j,j-1}I_{j-1,j-2}+I_{j-1,j-2}I^2_{j,j-1}-[2]\ I_{j,j-1}I_{j-1,j-2}I_{j,j-1}=-I_{j-1,j-2},\\ &I^2_{j-1,j-2}I_{j,j-1}+I_{j,j-1}I^2_{j-1,j-2}-[2]\ I_{j-1,j-2}I_{j,j-1}I_{j-1,j-2}=-I_{j,j-1},\\ &[I_{i,i-1},I_{j,j-1}]=0 \qquad \text{if} \quad |\, i-j\,|>1, \end{aligned} \tag{1}$$

where $q+q^{-1}\equiv[2]$, $q\in\mathbf{C}$, $q\neq 0,\pm1$. It is useful to introduce the generators

$$I^{\pm}_{k,l}\equiv[I_{l+1,l},I^{\pm}_{k,l+1}]_{q^{\pm1}}, \qquad\qquad k>l+1,\quad 1\le k,l\le n, \tag{2}$$

where $[X,Y]_{q^{\pm1}}\equiv q^{\pm1/2}XY-q^{\mp1/2}YX$ and $I^+_{k+1,k}\equiv I^-_{k+1,k}\equiv I_{k+1,k}$. If $q\to1$ ('classical' limit), the set of relations (1) reduce to those of $U(\mathrm{so}_n)$.

The algebra $U'_q(\mathrm{so}_n)$ can be embedded into quantum algebra $U_q(\mathrm{sl}_n)$, which is defined [1,2,15] as a complex associative algebra with the generating elements e_i, f_i, k_i, k_i^{-1}, $i=1,2,\ldots,n-1$, and defining relations

$$k_ik_i^{-1}=k_i^{-1}k_i=1,\quad k_ik_j=k_jk_i,\quad k_ie_jk_i^{-1}=q^{a_{ij}}e_j,\quad k_if_jk_i^{-1}=q^{-a_{ij}}f_j,$$

$$[e_i,e_j]=[f_i,f_j]=0,\quad |i-j|>1,\qquad [e_i,f_j]=\delta_{ij}\frac{k_i-k_i^{-1}}{q-q^{-1}},$$

$$e_i^2e_{i\pm1}-(q+q^{-1})e_ie_{i\pm1}e_i+e_{i\pm1}e_i^2=0,\quad f_i^2f_{i\pm1}-(q+q^{-1})f_if_{i\pm1}f_i+f_{i\pm1}f_i^2=0,$$

where $a_{ii}=2$, $a_{i,i\pm1}=-1$ and $a_{ij}=0$ for $|i-j|>1$. It is shown in [9,10], that the elements $\tilde{I}_{i+1,i}=f_i-q^{-1}k_ie_i$, $i=1,2,\ldots,n-1$, satisfy the relations (1) and define a homomorphism $U'_q(\mathrm{so}_n)\to U_q(\mathrm{sl}_n)$. Moreover, it is proved in [11] that this homomorphism is an embedding, that is, we may consider $U'_q(\mathrm{so}_n)$ as a subalgebra in $U_q(\mathrm{sl}_n)$.

The quantum algebra $U_q(\mathrm{sl}_n)$ possesses the Hopf structure. Comultiplication on generators of this algebra can be defined as

$$\Delta(e_i)=e_i\otimes k_i^{-1}+1\otimes e_i,\qquad \Delta(f_i)=f_i\otimes1+k_i\otimes f_i,\qquad \Delta(k_i)=k_i\otimes k_i.$$

Therefore, we obtain the coideal property of $U'_q(\mathrm{so}_n)$ embedded into $U_q(\mathrm{sl}_n)$:

$$\Delta(\tilde{I}_{i+1,i})=\tilde{I}_{i+1,i}\otimes1+k_i\otimes\tilde{I}_{i+1,i}.$$

Proposition 1. *The algebra $U'_q(\mathrm{so}_n)$ is a $U_q(\mathrm{sl}_n)$-comodule algebra with the coaction $\phi(I_{i+1,i})=\tilde{I}_{i+1,i}\otimes1+k_i\otimes I_{i+1,i}$. If one embeds $U'_q(\mathrm{so}_n)$ into $U_q(\mathrm{sl}_n)$, coaction ϕ reduces to comultiplication Δ of $U_q(\mathrm{sl}_n)$.*

Proof. This proposition can be verified by direct calculation. □

In particular, Proposition 1 claims that ϕ is a homomorphism from $U'_q(\mathrm{so}_n)$ into $U_q(\mathrm{sl}_n)\otimes U'_q(\mathrm{so}_n)$. This comodule structure can be used to introduce the tensor product of vector and arbitrary representations T of $U'_q(\mathrm{so}_n)$.

Let T be a representation of $U'_q(\mathrm{so}_n)$ on the linear space $\mathcal{V}$ with the basis $\{v_\alpha\}$ and $\mathcal{V}_{\mathbf{1}}$ be the n-dimensional linear space with the basis $\{v_k\}$, $k = 1, 2, \ldots, n$, and $\mathcal{V}^\otimes \equiv \mathcal{V}_{\mathbf{1}} \otimes \mathcal{V}$.

Proposition 2. *The map $T^\otimes$ from $U'_q(\mathrm{so}_n)$ to $\mathrm{End}\,\mathcal{V}^\otimes$ given by the formulas*

$$T^\otimes(I_{j,j-1})\,(v_{j-1}\otimes v_\alpha) = q\ v_{j-1}\otimes T(I_{j,j-1})v_\alpha - q^{1/2}\ v_j\otimes v_\alpha, \tag{3}$$

$$T^\otimes(I_{j,j-1})\,(v_{j}\otimes v_\alpha) = q^{-1}\ v_{j}\otimes T(I_{j,j-1})v_\alpha + q^{-1/2}\ v_{j-1}\otimes v_\alpha, \tag{4}$$

$$T^\otimes(I_{j,j-1})\,(v_{k}\otimes v_\alpha) = v_{k}\otimes T(I_{j,j-1})v_\alpha, \qquad j\neq k,\ j-1\neq k \tag{5}$$

defines a representation of $U'_q(\mathrm{so}_n)$ on the space $\mathcal{V}^\otimes$.

Proof. Let us define representation $\mathcal{T}_{\mathbf{1}}$ of $U_q(\mathrm{sl}_n)$ on the space $\mathcal{V}_{\mathbf{1}}$ by the formulas

$$\begin{array}{l} \mathcal{T}_{\mathbf{1}}(e_i)\,v_k = -q^{-1/2}\delta_{i+1,k}v_{k-1}, \quad \mathcal{T}_{\mathbf{1}}(f_i)\,v_k = -q^{1/2}\delta_{i,k}v_{k+1}, \\ \mathcal{T}_{\mathbf{1}}(k_i)\,v_k = q^{\delta_{i,k}-\delta_{i+1,k}}v_k. \end{array} \tag{6}$$

It is easy to verify that this representation is the vector representation (that is, representation with the highest weight $(1, 0, \ldots, 0)$). The action formulas (6) imply

$$\mathcal{T}_{\mathbf{1}}(\tilde{I}_{i+1,i})\,v_k = -q^{1/2}\delta_{i,k}v_{k+1} + q^{-1/2}\delta_{i+1,k}v_{k-1}. \tag{7}$$

This representation of $U'_q(\mathrm{so}_n)$ is equivalent to the classical type representation $T_{\mathbf{m}_n}$ with $\mathbf{m}_n = (1, 0, \ldots, 0)$, that is, the vector representation (see next section). Hence, similarly to the classical case, the restriction of the vector representation of $U_q(\mathrm{sl}_n)$ onto $U'_q(\mathrm{so}_n)$ is the vector representation of $U'_q(\mathrm{so}_n)$. This proposition immediately follows from Proposition 1 and formula (7), if one takes $T^\otimes = (\mathcal{T}_1\otimes T)\circ\phi$. □

In the case when T is the trivial representation of $U'_q(\mathrm{so}_n)$ given by formulas $T(a) = 0$, $a\in U'_q(\mathrm{so}_n)$, $a\neq 1$, Proposition 2 gives us a representation on the space $\mathcal{V}_{\mathbf{1}}\sim\mathcal{V}^\otimes$. We denote this representation by $T_{\mathbf{1}}$.

$$T_{\mathbf{1}}(I_{j,j-1})\,v_k = -q^{1/2}\delta_{k,j-1}v_j + q^{-1/2}\delta_{k,j}v_{j-1}.$$

The representations $T_{\mathbf{1}}$ and $T_{\mathbf{m}_n}$, $\mathbf{m}_n = (1, 0, \ldots, 0)$ (see next section), are equivalent.

In the limit $q\to 1$, Proposition 2 defines the representation which is the tensor product of the vector and some arbitrary representation of the Lie algebra so_n. On the base of these two arguments, we shall also use the notion $T^\otimes \equiv T_{\mathbf{1}}\otimes T$.

3. Finite dimensional classical type representations of $U'_q(\mathrm{so}_n)$

In this section we describe (in the framework of Gel'fand–Tsetlin formalism) irreducible finite-dimensional representation of the algebra $U'_q(\mathrm{so}_n)$, which are q-deformations of the finite-dimensional irreducible representations of the Lie algebra so_n. They are given

by sets $\mathbf{m}_n$ consisting of $\lfloor n/2 \rfloor$ numbers $m_{1,n}$, $m_{2,n}, \ldots$, $m_{\lfloor n/2 \rfloor,n}$ (here $\lfloor n/2 \rfloor$ denotes integral part of $n/2$) which are all integral or all half-integral and satisfy the dominance conditions

$$\begin{array}{l} m_{1,2p+1} \geq m_{2,2p+1} \geq \ldots \geq m_{p,2p+1} \geq 0, \\ m_{1,2p} \geq m_{2,2p} \geq \ldots \geq m_{p-1,2p} \geq |m_{p,2p}| \end{array} \tag{8}$$

for $n = 2p+1$ and $n = 2p$, respectively. These representations are denoted by $T_{\mathbf{m}_n}$. For a basis in a representation space $\mathcal{V}_{\mathbf{m}_n}$ we take the q-analogue of Gel'fand–Tsetlin basis which is obtained by successive reduction of the representation $T_{\mathbf{m}_n}$ to the subalgebras $U_q'(\mathrm{so}_{n-1})$, $U_q'(\mathrm{so}_{n-2})$, $\cdots$, $U_q'(\mathrm{so}_3)$, $U_q'(\mathrm{so}_2) \equiv U(\mathrm{so}_2)$. As in the classical case, its elements are labelled by Gel'fand–Tsetlin tableaux

$$\{\xi_n\} \equiv \{\mathbf{m}_n, \xi_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \xi_{n-2}\} \equiv \cdots \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \ldots, \mathbf{m}_2\}, \tag{9}$$

where the components of $\mathbf{m}_k$ and $\mathbf{m}_{k-1}$ satisfy the "betweenness" conditions

$$m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq \ldots \geq m_{p,2p+1} \geq m_{p,2p} \geq -m_{p,2p+1},$$
$$m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq \ldots \geq m_{p-1,2p-1} \geq |m_{p,2p}|.$$

The basis element defined by tableau $\{\xi_n\}$ is denoted as $|\xi_n\rangle$. We suppose that the representation space $\mathcal{V}_{\mathbf{m}_n}$ is a Hilbert space and vectors $|\xi_n\rangle$ are orthonormal. It is convenient to introduce the so-called l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \qquad l_{j,2p} = m_{j,2p} + p - j \tag{10}$$

for the numbers $m_{i,k}$. The operator $T_{\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\mathbf{m}_n}$ of $U_q'(\mathrm{so}_n)$ acts upon Gel'fand–Tsetlin basis elements, labeled by (9), as

$$T_{\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \sum_{j=1}^{p} A_{2p}^{j}(\xi_n)|(\xi_n)_{2p}^{+j}\rangle - \sum_{j=1}^{p} A_{2p}^{j}((\xi_n)_{2p}^{-j})|(\xi_n)_{2p}^{-j}\rangle \tag{11}$$

and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ of the representation $T_{\mathbf{m}_n}$ acts as

$$\begin{array}{l} T_{\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1} B_{2p-1}^{j}(\xi_n)|(\xi_n)_{2p-1}^{+j}\rangle \\ -\sum_{j=1}^{p-1} B_{2p-1}^{j}((\xi_n)_{2p-1}^{-j})|(\xi_n)_{2p-1}^{-j}\rangle + \mathrm{i}\, C_{2p-1}(\xi_n)|\xi_n\rangle, \\ T_{\mathbf{m}_n}(I_{21})|\xi_n\rangle = \mathrm{i}\,[l_{12}]|\xi_n\rangle. \end{array} \tag{12}$$

In these formulas, $(\xi_n)_k^{\pm j}$ means the tableau (9) in which j-th component $m_{j,k}$ in $\mathbf{m}_k$ is replaced by $m_{j,k} \pm 1$. The coefficients A_{2p}^{j}, B_{2p-1}^{j}, C_{2p-1} in (11) and (12) are given by the expressions

$$A_{2p}^{j}(\xi_n) = \left(\frac{[l_{j,2p}][l_{j,2p}+1]}{[2l_{j,2p}][2l_{j,2p}+2]} \right)^{\frac{1}{2}} \hat{A}_2^j p\ , \tag{13}$$

$$B_{2p-1}^{j}(\xi_n) = \frac{\hat{B}_{2p-1}^{j}(\xi_n)}{[l_{j,2p-1}]([2l_{j,2p-1}+1][2l_{j,2p-1}-1])^{\frac{1}{2}}}, \tag{14}$$

$$\hat{A}^j_{2p} = \left(\frac{\prod_{i=1}^{p}[l_{i,2p+1}+l_{j,2p}][l_{i,2p+1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}][l_{i,2p}-l_{j,2p}]} \right.$$

$$\left. \times \frac{\prod_{i=1}^{p-1}[l_{i,2p-1}+l_{j,2p}][l_{i,2p-1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}+1][l_{i,2p}-l_{j,2p}-1]} \right)^{\frac{1}{2}} \tag{15}$$

and

$$\hat{B}^j_{2p-1}(\xi_n) = \left(\frac{\prod_{i=1}^{p}[l_{i,2p}+l_{j,2p-1}][l_{i,2p}-l_{j,2p-1}]}{\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}][l_{i,2p-1}-l_{j,2p-1}]} \right.$$

$$\left. \times \frac{\prod_{i=1}^{p-1}[l_{i,2p-2}+l_{j,2p-1}][l_{i,2p-2}-l_{j,2p-1}]}{\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}-1][l_{i,2p-1}-l_{j,2p-1}-1]} \right)^{\frac{1}{2}}, \tag{16}$$

$$C_{2p-1}(\xi_n) = \frac{\prod_{i=1}^{p}[l_{i,2p}]\prod_{i=1}^{p-1}[l_{i,2p-2}]}{\prod_{i=1}^{p-1}[l_{i,2p-1}][l_{i,2p-1}-1]}, \tag{17}$$

where numbers in square brackets mean q-numbers defined by $[a] := (q^a - q^{-a})/(q - q^{-1})$.

4. Finite dimensional nonclassical type representations of $U'_q(\mathrm{so}_n)$

The representations of the previous section are called representations of the classical type, because at $q \to 1$ the operators $T_{\mathbf{m}_n}(I_{j,j-1})$ turn into the corresponding operators $T_{\mathbf{m}_n}(I_{j,j-1})$ for irreducible finite dimensional representations with highest weights $\mathbf{m}_n$ of the Lie algebra so_n.

The algebra $U'_q(\mathrm{so}_n)$ also has irreducible finite dimensional representations T of nonclassical type, that is, such that the operators $T(I_{j,j-1})$ have no classical limit $q \to 1$. They are given (see [16]) by sets $\epsilon := (\epsilon_2, \epsilon_3, \cdots, \epsilon_n)$, $\epsilon_i = \pm 1$, and by sets $\mathbf{m}_n$ consisting of $\lfloor n/2 \rfloor$ **half-integral** numbers $m_{1,n}, m_{2,n}, \ldots, m_{\lfloor n/2 \rfloor,n}$ (here $\lfloor n/2 \rfloor$ denotes integral part of $n/2$) that satisfy the dominance conditions

$$m_{1,n} \geq m_{2,n} \geq \ldots \geq m_{\lfloor n/2 \rfloor,n} \geq 1/2. \tag{18}$$

These representations are denoted by $T_{\epsilon,\mathbf{m}_n}$.

For a basis in the representation space $\tilde{\mathcal{V}}_{\mathbf{m}_n}$ we use the analogue of the basis of the previous section. Its elements are labeled by tableaux

$$\{\xi_n\} \equiv \{\mathbf{m}_n, \xi_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \xi_{n-2}\} \equiv \cdots \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \ldots, \mathbf{m}_2\}, \tag{19}$$

where the components of $\mathbf{m}_k$ and $\mathbf{m}_{k-1}$ satisfy the "betweenness" conditions

$$\begin{aligned} &m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq \ldots \geq m_{p,2p+1} \geq m_{p,2p} \geq 1/2, \\ &m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq \ldots \geq m_{p-1,2p-1} \geq m_{p,2p}. \end{aligned}$$

The basis element defined by tableau $\{\xi_n\}$ is denoted as $|\xi_n\rangle$. We suppose that the representation space $\tilde{\mathcal{V}}_{\mathbf{m}_n}$ is a Hilbert space and vectors $|\xi_n\rangle$ are orthonormal. It is convenient to introduce the l-coordinates as in (10).

The operator $T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\epsilon,\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ acts upon basis elements, labeled by (19), by the formula

$$T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \delta_{m_{p,2p},1/2}\,\frac{\epsilon_{2p+1}}{q^{1/2}-q^{-1/2}}D_{2p}(\xi_n)|\xi_n\rangle$$

$$+\sum_{j=1}^{p}\tilde{A}^j_{2p}(\xi_n)|(\xi_n)^{+j}_{2p}\rangle - \sum_{j=1}^{p}\tilde{A}^j_{2p}((\xi_n)^{-j}_{2p})|(\xi_n)^{-j}_{2p}\rangle, \tag{20}$$

where the summation in the last sum must be from 1 to $p-1$ if $m_{p,2p}=1/2$, and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ of the representation $T_{\mathbf{m}_n}$ acts as

$$T_{\epsilon,\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1}\tilde{B}^j_{2p-1}(\xi_n)|(\xi_n)^{+j}_{2p-1}\rangle$$

$$-\sum_{j=1}^{p-1}\tilde{B}^j_{2p-1}((\xi_n)^{-j}_{2p-1})|(\xi_n)^{-j}_{2p-1}\rangle + \epsilon_{2p}\tilde{C}_{2p-1}(\xi_n)|\xi_n\rangle, \tag{21}$$

$$T_{\epsilon,\mathbf{m}_n}(I_{21})|\xi_n\rangle = \epsilon_2[l_{12}]_+|\xi_n\rangle,$$

where $[a]_+ := (q^a+q^{-a})/(q-q^{-1})$. In these formulas, $(\xi_n)^{\pm j}_k$ means the tableau (19) in which j-th component $m_{j,k}$ in $\mathbf{m}_k$ is replaced by $m_{j,k}\pm 1$. Matrix elements $\tilde{A}^j_{2p}$ and $\tilde{B}^j_{2p-1}$ are defined using formulas (15) and (16):

$$\tilde{A}^j_{2p}(\xi_n) = \frac{\hat{A}^j_{2p}(\xi_n)}{((q^{l_{j,2p}}-q^{-l_{j,2p}})(q^{l_{j,2p}+1}-q^{-l_{j,2p}-1}))^{\ frac12}}$$

$$\tilde{B}^j_{2p-1}(\xi_n) = \frac{\hat{B}^j_{2p-1}(\xi_n)}{[l_{j,2p-1}]_+([2l_{j,2p-1}+1][2l_{j,2p-1}-1])^{\frac12}},$$

$$\tilde{C}_{2p-1}(\xi_n) = \frac{\prod_{s=1}^{p}[l_{s,2p}]_+\prod_{s=1}^{p-1}[l_{s,2p-2}]_+}{\prod_{s=1}^{p-1}[l_{s,2p-1}]_+[l_{s,2p-1}-1]_+}.$$

$$D_{2p}(\xi_n) = \frac{\prod_{i=1}^{p}[l_{i,2p+1}-\frac12]\prod_{i=1}^{p-1}[l_{i,2p-1}-\frac12]}{\prod_{i=1}^{p-1}[l_{i,2p}+\frac12][l_{i,2p}-\frac12]}.$$

5. Decomposition of representations $T_{\mathbf{1}}\otimes T_{\mathbf{m}_3}$ of the algebra $U'_q(\mathrm{so}_3)$

In this and in the next sections, we consider the decomposition of representations $T^{\otimes}\equiv T_{\mathbf{1}}\otimes T_{\mathbf{m}_n}$ into irreducible constituents of the algebra $U'_q(\mathrm{so}_n)$. In this section, we restrict ourselves to the case $n=2,3$.

First, we consider the case of the algebra $U'_q(\mathrm{so}_2)\equiv U(\mathrm{so}_2)$. This algebra has representations T_m, $m\equiv m_{12}$, $m\in\frac12\mathbf{Z}$, of the *classical type* acting on one-dimensional spaces with basis vectors $|m\rangle$, and $T_m(I_{21})|m\rangle = \mathrm{i}[m]|m\rangle$. Then

$$T^{\otimes}(I_{21})(v_1\otimes|m\rangle) = \mathrm{i}q[m]v_1\otimes|m\rangle - q^{1/2}v_2\otimes|m\rangle,$$

$$T^{\otimes}(I_{21})(v_2\otimes|m\rangle) = \mathrm{i}q^{-1}[m]v_2\otimes|m\rangle + q^{-1/2}v_1\otimes|m\rangle.$$

This representation is reducible. We introduce the vectors

$$v_{\pm}^{(m)} = \mp\mathrm{i}q^{-1/2\pm m}v_1 + v_2. \tag{22}$$

Then the vectors $|m\pm 1\rangle^{\otimes} := v_{\pm}^{(m)}\otimes|m\rangle$ are eigenvectors of $T^{\otimes}(I_{21})$: $T^{\otimes}(I_{21})|m\pm 1\rangle^{\otimes} = \mathrm{i}[m\pm 1]|m\pm 1\rangle^{\otimes}$. This fact can be easy verified by direct calculation using the definition of q-numbers. Thus, we have decomposition $T^{\otimes}\equiv T_{\mathbf{1}}\otimes T_m = T_{m+1}\oplus T_{m-1}$.

Now, we consider the case of the algebra $U_q'(\mathrm{so}_3)$. This algebra has representations T_l, $\mathbf{m}_3\equiv(m_{13})\equiv(l)$, $l\in\{0,\,1/2,\,1,\,3/2,\ldots\}$, of the *classical type* acting on the spaces $\mathcal{V}_l$ with the basis vectors $|l,m\rangle$, $(m\equiv m_{12})$, $m=-l,-l+1,\ldots,l$:

$$T_l(I_{21})|l,m\rangle = \mathrm{i}[m]|l,m\rangle, \qquad T_l(I_{32})|l,m\rangle = A_{l,m}|l,m+1\rangle - A_{l,m-1}|l,m-1\rangle,$$

where $A_{l,m} = d_m([l-m][l+m+1])^{1/2}$, $d_m = ([m][m+1]/([2m][2m+2]))^{1/2}$. Let us consider the vectors

$$|l',m\rangle^{\otimes} := \alpha_{l,m}^{(l')}v_{+}^{(m-1)}\otimes|l,m-1\rangle + \beta_{l,m}^{(l')}v_3\otimes|l,m\rangle + \gamma_{l,m}^{(l')}v_{-}^{(m+1)}\otimes|l,m+1\rangle, \tag{23}$$

where $m=-l',-l'+1,\ldots,l'$, and

$$l'=l+1,l,l-1 \quad \text{if} \quad l\ge 1; \qquad l'=3/2,1/2 \quad \text{if} \quad l=1/2; \qquad l'=1 \quad \text{if} \quad l=0.$$

The vectors $v_{\pm}^{(m)}$ in (23) are defined in (22) and

$$\alpha_{l,m}^{(l+1)} = q^{l-m+1/2}d_{m-1}([l+m][l+m+1])^{1/2},$$

$$\beta_{l,m}^{(l+1)} = ([l-m+1][l+m+1])^{1/2},$$

$$\gamma_{l,m}^{(l+1)} = -q^{l+m+1/2}d_m([l-m][l-m+1])^{1/2},$$

$$\alpha_{l,m}^{(l)} = -q^{-m-1/2}d_{m-1}([l+m][l-m+1])^{1/2},$$

$$\beta_{l,m}^{(l)} = [m],$$

$$\gamma_{l,m}^{(l)} = -q^{m-1/2}d_m([l-m][l+m+1])^{1/2},$$

$$\alpha_{l,m}^{(l-1)} = -q^{-l-m-1/2}d_{m-1}([l-m][l-m+1])^{1/2},$$

$$\beta_{l,m}^{(l-1)} = ([l-m][l+m])^{1/2},$$

$$\gamma_{l,m}^{(l-1)} = q^{-l+m-1/2}d_m([l+m][l+m+1])^{1/2}.$$

From the case of $U_q'(\mathrm{so}_2)$, it is easy to see that $T^{\otimes}(I_{21})|l',m\rangle^{\otimes} = \mathrm{i}[m]|l',m\rangle^{\otimes}$. One can show by direct calculation that $T^{\otimes}(I_{32})|l',m\rangle^{\otimes} = A_{l',m}|l',m+1\rangle^{\otimes} - A_{l',m-1}|l',m-1\rangle^{\otimes}$. It means that the vectors $|l',m\rangle^{\otimes}$ at fixed l' span a subspace in $\mathcal{V}^{\otimes}$, which is

invariant and irreducible under the action of $T^{\otimes}(a)$, $a\in U'_q(\mathrm{so}_3)$. The corresponding subrepresentation is equivalent to $T_{l'}$. Comparing the dimensions of $T_{l'}$ with dimension of $T^{\otimes}$, we conclude that $T^{\otimes}=T_{l+1}\oplus T_l\oplus T_{l-1}$, if $l\ge 1$; $T^{\otimes}=T_{3/2}\oplus T_{1/2}$, if $l=1/2$; $T^{\otimes}=T_1$, if $l=0$. Let us remind that $T_l\equiv T_{\mathbf{m}_3}$, $m_{13}\equiv l$. The numbers $\alpha^{(l')}_{l,m}$, $\beta^{(l')}_{l,m}$ and $\gamma^{(l')}_{l,m}$ are Clebsch–Gordan coefficients of these decompositions.

6. Decomposition of $T_{\mathbf{1}}\otimes T_{\mathbf{m}_n}$ of the algebra $U'_q(\mathrm{so}_n)$, $n\ge 4$

In this section, we consider the decomposition of the representations $T^{\otimes}\equiv T_{\mathbf{1}}\otimes T_{\mathbf{m}_n}$ of algebra $U'_q(\mathrm{so}_n)$, $n\ge 4$, into irreducible constituents. All the results of this section are obtained in [12]. As shown there, this decomposition has the form

$$T^{\otimes}=\bigoplus_{\mathbf{m}'_n\in\mathcal{S}(\mathbf{m}_n)} T_{\mathbf{m}'_n},\tag{24}$$

where

$$\mathcal{S}(\mathbf{m}_{2p+1})=\bigcup_{j=1}^{p}\{\mathbf{m}^{+j}_{2p+1}\}\cup\bigcup_{j=1}^{p}\{\mathbf{m}^{-j}_{2p+1}\}\cup\{\mathbf{m}_{2p+1}\},\tag{25}$$

$$\mathcal{S}(\mathbf{m}_{2p})=\bigcup_{j=1}^{p}\{\mathbf{m}^{+j}_{2p}\}\cup\bigcup_{j=1}^{p}\{\mathbf{m}^{-j}_{2p}\}.\tag{26}$$

By $\mathbf{m}^{\pm j}_n$ we mean here the set $\mathbf{m}_n$ with $m_{j,n}$ replaced by $m_{j,n}\pm 1$, respectively. If some $\mathbf{m}^{\pm j}_n$ is not dominant (8), then the corresponding $\mathbf{m}^{\pm j}_n$ must be omitted. If $m_{p,2p+1}=0$ then $\mathbf{m}_{2p+1}$ in right-hand side of (25) also must be omitted. For decomposition (24) of the representation $T^{\otimes}$, there correspond the decomposition of carrier space:

$$\mathcal{V}^{\otimes}\equiv\mathcal{V}_{\mathbf{1}}\otimes\mathcal{V}_{\mathbf{m}_n}=\bigoplus_{\mathbf{m}'_n\in\mathcal{S}(\mathbf{m}_n)}\mathcal{V}_{\mathbf{m}'_n}.\tag{27}$$

In order to give this decomposition in an explicit form, we change the basis $\{v_k\otimes|\xi_n\rangle\}$, $k=1,2,\dots,n$, in $\mathcal{V}^{\otimes}$ to $\{v_k\otimes|\xi_n\rangle\}$, $k=+,-,3,\dots,n$, by replacing (for every fixed $\{\xi_n\}=\{\mathbf{m}_n,\mathbf{m}_{n-1},\dots,\mathbf{m}_3,\mathbf{m}_2\}$) two basis vectors $v_1\otimes|\xi_n\rangle$ and $v_2\otimes|\xi_n\rangle$ by $v_+^{(m_{12})}\otimes|\xi_n\rangle$ and $v_-^{(m_{12})}\otimes|\xi_n\rangle$ (see (22)). From now on, we shall omit the index (m_{12}) in the notion of the basis vectors $v_\pm^{(m_{12})}\otimes|\xi_n\rangle$, supposing that it is equal to m_{12}-component of the corresponding Gel'fand–Tsetlin tableaux $\{\xi_n\}$.

We introduce the vectors (where $\{\xi'_n\}=\{\mathbf{m}'_n,\mathbf{m}'_{n-1},\dots,\mathbf{m}'_3,\mathbf{m}'_2\}$)

$$|\mathbf{m}'_n,\xi_{n-1}\rangle^{\otimes}:=\sum_k\sum_{|\mathbf{m}_n,\xi'_{n-1}\rangle\in\mathcal{V}_{\mathbf{m}_n}}(k,(\mathbf{m}_n,\xi'_{n-1})|(\mathbf{m}'_n,\xi_{n-1}))\,v_k\otimes|\mathbf{m}_n,\xi'_{n-1}\rangle\tag{28}$$

in the space $\mathcal{V}^{\otimes}$, where k runs over the set $+,-,3,\dots,n$, and coefficients $(k,(\mathbf{m}_n,\xi'_{n-1})|(\mathbf{m}'_n,\xi_{n-1}))$ are Clebsch–Gordan coefficients (CGC's). Now we define these CGC's in an explicit form.

We put $(k,(\mathbf{m}_n,\xi'_{n-1})|(\mathbf{m}'_n,\xi_{n-1}))=0$ if one of the conditions

1) $\mathbf{m}'_n\notin\mathcal{S}(\mathbf{m}_n)$,
2) $\mathbf{m}_s\notin\mathcal{S}(\mathbf{m}'_s), s=n-1,\dots,k,\quad k\ge 3$,
3) $\mathbf{m}_s\notin\mathcal{S}(\mathbf{m}'_s), s=n-1,\dots,3,\quad k=+,-$,
4) $\xi'_{k-1}\ne\xi_{k-1}, k=3,4,\dots,n$,
5) $m_{12}\ne m'_{12}+1, k=+$,
6) $m_{12}\ne m'_{12}-1, k=-$.

is fulfilled. The nonzero CGC for $k=n$ are:

$$
\begin{aligned}
&(2p+1,(\mathbf{m}_{2p+1},\xi_{2p})|(\mathbf{m}^{+j}_{2p+1},\xi_{2p}))=\Big(\prod_{r=1}^{p}[l_{j,2p+1}+l_{r,2p}][l_{j,2p+1}-l_{r,2p}]\Big)^{\frac12},\\
&(2p+1,(\mathbf{m}_{2p+1},\xi_{2p})|(\mathbf{m}_{2p+1},\xi_{2p}))=\prod_{r=1}^{p}[l_{r,2p}],\\
&(2p+1,(\mathbf{m}_{2p+1},\xi_{2p})|(\mathbf{m}^{-j}_{2p+1},\xi_{2p}))=\Big(\prod_{r=1}^{p}[l_{j,2p+1}+l_{r,2p}-1][l_{j,2p+1}-l_{r,2p}-1]\Big)^{\frac12},
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&(2p,(\mathbf{m}_{2p},\xi_{2p-1})|(\mathbf{m}^{+j}_{2p},\xi_{2p-1}))=\Big(\prod_{r=1}^{p-1}[l_{j,2p}+l_{r,2p-1}][l_{j,2p}-l_{r,2p-1}+1]\Big)^{\frac12},\\
&(2p,(\mathbf{m}_{2p},\xi_{2p-1})|(\mathbf{m}^{-j}_{2p},\xi_{2p-1}))=\Big(\prod_{r=1}^{p-1}[l_{j,2p}+l_{r,2p-1}-1][l_{j,2p}-l_{r,2p-1}]\Big)^{\frac12}.
\end{aligned}
\tag{30}
$$

(They are defined up to normalization, that is, multiplication of these CGC's by some constants will not spoil the following results.)

All the other CGC's can be found from just presented as follows:

$$
(k,\xi_n|\xi'_n)=q^{k-n}\frac{\langle\mathbf{m}_{n+1},\xi_n|T_{\mathbf{m}_{n+1}}(I^-_{n+1,k})|\mathbf{m}_{n+1},\xi'_n\rangle}{\langle\mathbf{m}_{n+1},\mathbf{m}_n,\xi_{n-1}|T_{\mathbf{m}_{n+1}}(I_{n+1,n})|\mathbf{m}_{n+1},\mathbf{m}'_n,\xi_{n-1}\rangle}
\times(n,(\mathbf{m}_n,\xi_{n-1})|(\mathbf{m}'_n,\xi_{n-1})),\tag{31}
$$

where the generators $I^-_{n+1,k}$ are defined in (2). If $k=+$ or $k=-$ in the left-hand side of (31), one must put $k=2$ in right-hand side. The set $\mathbf{m}_{n+1}$ must be chosen to give nonzero denominator in right-hand side of (31). Note that if $(n,(\mathbf{m}_n,\xi_{n-1})|(\mathbf{m}'_n,\xi_{n-1}))\ne 0$, one can always do such a choice, moreover, the resulting CGC will not depend on this particular choice. In the case $n=3$ we reobtain the CGC's for the algebra $U'_q(\mathrm{so}_3)$ (see section 5).

As shown in [12], the defined CGC's have the *factorization* property. This fact (in complete analogy with the classical case, see [13,14]) gives a possibility to present arbitrary CGC for the algebra $U'_q(\mathrm{so}_n)$ as a product of *scalar factors*.

Theorem 1. *The formulas for the action of the operators $T^{\otimes}(I_{k+1,k})$, $k=1,2,\dots,n-1$, on the vectors $|\mathbf{m}'_n,\xi_{n-1}\rangle^{\otimes}$ defined by (28) with CGC's defined by (29)–(31), coincide with the corresponding formulas (11)–(12) for the action of the operators $T_{\mathbf{m}'_n}(I_{k+1,k})$ on the GT basis vectors $|\mathbf{m}'_n,\xi_{n-1}\rangle$. We have the decomposition (24).*

7. Decomposition of representations $T_{\mathbf{1}}\otimes T_{\epsilon,\mathbf{m}_3}$ of the algebra $U'_q(\mathrm{so}_3)$

In this and in the next section, we consider the decomposition of representations $T^{\otimes}\equiv T_{\mathbf{1}}\otimes T_{\epsilon,\mathbf{m}_n}$ into irreducible constituents of the algebra $U'_q(\mathrm{so}_n)$. In this section, we restrict ourselves to the case $n=2,3$.

First, we consider the case of the algebra $U'_q(\mathrm{so}_2)\equiv U(\mathrm{so}_2)$. This algebra has representations $T_{\epsilon_2,m}$, $\epsilon_2=\pm1$, $m\equiv m_{12}$, $m\in\{1/2,3/2,\ldots\}$, acting on one-dimensional spaces with basis vectors $|m\rangle$, and $T_{\epsilon_2,m}(I_{21})|m\rangle=\epsilon_2[m]_+|m\rangle$. Then the representation $T^{\otimes}\equiv T_{\mathbf{1}}\otimes T_{\epsilon_2,m}$ is two-dimensional and reducible. We introduce the vectors

$$v_{\pm}^{(\epsilon_2,m)}=-\epsilon_2 q^{-1/2\pm m}v_1+v_2. \tag{32}$$

Then the vectors $|m\pm1\rangle^{\otimes}:=v_{\pm}^{(\epsilon_2,m)}\otimes|m\rangle$ are eigenvectors of $T^{\otimes}(I_{21})$: $T^{\otimes}(I_{21})|m\pm1\rangle^{\otimes}=\epsilon_2[m\pm1]_+|m\pm1\rangle^{\otimes}$. This fact can be easy verified by direct calculation using the definition of q-numbers. Thus, we have decomposition $T^{\otimes}\equiv T_{\mathbf{1}}\otimes T_{\epsilon,m}=T_{\epsilon_2,m+1}\oplus T_{\epsilon_2,m-1}$, if $m\geq3/2$, and $T^{\otimes}\equiv T_{\mathbf{1}}\otimes T_{\epsilon_2,1/2}=T_{\epsilon_2,3/2}\oplus T_{\epsilon_2,1/2}$.

Now, we consider the case of the algebra $U'_q(\mathrm{so}_3)$. This algebra has four classes of representations of *nonclassical type* $T_{\epsilon,l}$, $\epsilon=\{\epsilon_2,\epsilon_3\}$, $\epsilon_i\in\{\pm1\}$, $\mathbf{m}_3\equiv(m_{13})\equiv(l)$, $l\in\{1/2,3/2,5/2,\ldots\}$, acting on the spaces $\mathcal{V}_l$ with the basis vectors $|l,m\rangle$, $(m\equiv m_{12})$, $m=1/2,3/2,\ldots,l$:

$$T_{\epsilon,l}(I_{21})|l,m\rangle=\epsilon_2[m]_+|l,m\rangle,$$
$$T_{\epsilon,l}(I_{32})|l,m\rangle=\tilde{A}_{l,m}|l,m+1\rangle-\tilde{A}_{l,m-1}|l,m-1\rangle,\qquad \text{if } m\geq3/2,$$
$$T_{\epsilon,l}(I_{32})|l,1/2\rangle=\tilde{A}_{l,1/2}|l,3/2\rangle+\epsilon_3[1/2]_+[l+1/2]|l,1/2\rangle,$$

where $\tilde{A}_{l,m}=\tilde{d}_m([l-m][l+m+1])^{1/2}$, $\tilde{d}_m=((q^m-q^{-m})(q^{m+1}-q^{-m-1}))^{-1/2}$. Let us consider the vectors

$$|l',m\rangle^{\otimes}:=\tilde{\alpha}_{l,m}^{(l')}v_{+}^{(\epsilon_2,m-1)}\otimes|l,m-1\rangle+\tilde{\beta}_{l,m}^{(l')}v_3\otimes|l,m\rangle+\tilde{\gamma}_{l,m}^{(l')}v_{-}^{(\epsilon_2,m+1)}\otimes|l,m+1\rangle, \tag{33}$$

where $m=3/2,5/2,\ldots,l'$, and

$$l'=l+1,l,l-1\quad\text{if}\ \ l\geq3/2;\qquad l'=3/2,1/2\quad\text{if}\ \ l=1/2.$$

If $m=1/2$, we should replace $|l,-1/2\rangle$ by $|l,1/2\rangle$ in right-hand side of (33).

The vectors $v_{\pm}^{(\epsilon_2,m)}$ in (33) are defined in (32) and

$$\tilde{\alpha}_{l,m}^{(l+1)}=q^{l-m+1/2}\tilde{d}_{m-1}([l+m][l+m+1])^{1/2},\qquad m\neq1/2$$
$$\tilde{\beta}_{l,m}^{(l+1)}=([l-m+1][l+m+1])^{1/2},$$
$$\tilde{\gamma}_{l,m}^{(l+1)}=-q^{l+m+1/2}\tilde{d}_m([l-m][l-m+1])^{1/2},$$
$$\tilde{\alpha}_{l,m}^{(l)}=q^{-m-1/2}\tilde{d}_{m-1}([l+m][l-m+1])^{1/2},\qquad m\neq1/2$$
$$\tilde{\beta}_{l,m}^{(l)}=[m]_+,$$

$$\tilde{\gamma}^{(l)}_{l,m} = -q^{m-1/2}\tilde{d}_m([l-m][l+m+1])^{1/2},$$
$$\tilde{\alpha}^{(l-1)}_{l,m} = -q^{-l-m-1/2}\tilde{d}_{m-1}([l-m][l-m+1])^{1/2}, \qquad m \neq 1/2$$
$$\tilde{\beta}^{(l-1)}_{l,m} = ([l-m][l+m])^{1/2},$$
$$\tilde{\gamma}^{(l-1)}_{l,m} = q^{-l+m-1/2}\tilde{d}_m([l+m][l+m+1])^{1/2},$$
$$\tilde{\alpha}^{(l+1)}_{l,1/2} = -q^{l}[1/2]_+\epsilon_3([l+1/2][l+3/2])^{1/2},$$
$$\tilde{\alpha}^{(l+1)}_{l,1/2} = -q^{-1}[1/2]_+\epsilon_3[l+1/2],$$
$$\tilde{\alpha}^{(l-1)}_{l,1/2} = q^{-l-1}[1/2]_+\epsilon_3([l-1/2][l+1/2])^{1/2}.$$

From the case of $U'_q(\mathrm{so}_2)$, it is easy to see that $T^{\otimes}(I_{21})|l',m\rangle^{\otimes} = \epsilon_2[m]_+|l',m\rangle^{\otimes}$. One can show by direct calculation that the operator $T^{\otimes}(I_{32})$ acts on the set of vectors $|l',m\rangle^{\otimes}$ at some fixed l' as operator $T_{\epsilon,l'}(I_{32})$ acts on the Gel'fand–Tsetlin basis vectors $|l',m\rangle$. It means that the vectors $|l',m\rangle^{\otimes}$ at fixed l' span a subspace in $\mathcal{V}^{\otimes}$, which is invariant and irreducible under the action of $T^{\otimes}(a)$, $a \in U'_q(\mathrm{so}_3)$. The corresponding subrepresentation is equivalent to $T_{\epsilon,l'}$. Comparing the dimensions of $T_{\epsilon,l'}$ with dimension of $T^{\otimes}$, we conclude that $T^{\otimes} = T_{\epsilon,l+1} \oplus T_{\epsilon,l} \oplus T_{\epsilon,l-1}$, if $l \geq 3/2$; $T^{\otimes} = T_{\epsilon,3/2} \oplus T_{\epsilon,1/2}$, if $l = 1/2$. Let us remind that $T_{\epsilon,l} \equiv T_{\epsilon,\mathbf{m}_3}$, $m_{13} \equiv l$. The numbers $\tilde{\alpha}^{(l')}_{l,m}$, $\tilde{\beta}^{(l')}_{l,m}$ and $\tilde{\gamma}^{(l')}_{l,m}$ are Clebsch–Gordan coefficients of these decompositions.

8. Decomposition of $T_{\mathbf{1}} \otimes T_{\epsilon,\mathbf{m}_n}$ of the algebra $U'_q(\mathrm{so}_n)$, $n \geq 4$

In this section, we describe the decomposition of the representations $T^{\otimes} \equiv T_{\mathbf{1}} \otimes T_{\epsilon,\mathbf{m}_n}$ of algebra $U'_q(\mathrm{so}_n)$, $n \geq 4$, into irreducible constituents. This decomposition has the form
$$T^{\otimes} = \bigoplus_{\mathbf{m}'_n \in \mathcal{S}(\mathbf{m}_n)} T_{\epsilon,\mathbf{m}'_n}, \tag{34}$$
where
$$\mathcal{S}(\mathbf{m}_{2p+1}) = \bigcup_{j=1}^{p}\{\mathbf{m}^{+j}_{2p+1}\} \cup \bigcup_{j=1}^{p}\{\mathbf{m}^{-j}_{2p+1}\} \cup \{\mathbf{m}_{2p+1}\}, \tag{35}$$
$$\mathcal{S}(\mathbf{m}_{2p}) = \bigcup_{j=1}^{p}\{\mathbf{m}^{+j}_{2p}\} \cup \bigcup_{j=1}^{p}\{\mathbf{m}^{-j}_{2p}\}. \tag{36}$$

By $\mathbf{m}^{\pm j}_n$ we mean here the set $\mathbf{m}_n$ with $m_{j,n}$ replaced by $m_{j,n} \pm 1$, respectively. If $m_{p,2p} = 1/2$, the element $\mathbf{m}^{-p}_{2p}$ in right-hand side of (36) must be replaced by $\mathbf{m}_{2p}$. If some $\mathbf{m}^{\pm j}_n$ is not dominant (18), then the corresponding $\mathbf{m}^{\pm j}_n$ must be omitted; in particular, if $m_{p,2p+1} = 1/2$, the element $\mathbf{m}^{-p}_{2p+1}$ must be omitted. Note, that the representation $T_{\mathbf{1}} \otimes T_{\epsilon,\mathbf{m}_n}$ decomposes into irreducible nonclassical type representations with the same set $\epsilon = (\epsilon_2, \epsilon_3, \ldots)$. For decomposition (34) of the representation $T^{\otimes}$, there correspond the decomposition of carrier space:
$$\mathcal{V}^{\otimes} \equiv \mathcal{V}_{\mathbf{1}} \otimes \tilde{\mathcal{V}}_{\epsilon,\mathbf{m}_n} = \bigoplus_{\mathbf{m}'_n \in \mathcal{S}(\mathbf{m}_n)} \tilde{\mathcal{V}}_{\epsilon,\mathbf{m}'_n}. \tag{37}$$

In order to give this decomposition in an explicit form, we change the basis $\{v_k \otimes |\xi_n\rangle\}$, $k = 1, 2, \dots, n$, in $\mathcal{V}^{\otimes}$ to $\{v_k \otimes |\xi_n\rangle\}$, $k = +, -, 3, \dots, n$, by replacing (for every fixed $\{\xi_n\} = \{\mathbf{m}_n, \mathbf{m}_{n-1}, \dots, \mathbf{m}_3, \mathbf{m}_2\}$) two basis vectors $v_1 \otimes |\xi_n\rangle$ and $v_2 \otimes |\xi_n\rangle$ by $v_+^{(\epsilon_2, m_{12})} \otimes |\xi_n\rangle$ and $v_-^{(\epsilon_2, m_{12})} \otimes |\xi_n\rangle$ (see (32)). From now on, we shall omit the index (ϵ_2, m_{12}) in the notion of the basis vectors $v_\pm^{(\epsilon_2, m_{12})} \otimes |\xi_n\rangle$, supposing that it contains m_{12}-component of the corresponding Gel'fand–Tsetlin tableaux $\{\xi_n\}$.

We introduce the vectors (where $\{\xi'_n\} = \{\mathbf{m}'_n, \mathbf{m}'_{n-1}, \dots, \mathbf{m}'_3, \mathbf{m}'_2\}$)

$$|\mathbf{m}'_n, \xi_{n-1}\rangle^{\otimes} := \sum_k \sum_{|\mathbf{m}_n, \xi'_{n-1}\rangle \in \mathcal{V}_{\mathbf{m}_n}} (k, (\mathbf{m}_n, \xi'_{n-1}) | (\mathbf{m}'_n, \xi_{n-1}); \epsilon)\, v_k \otimes |\mathbf{m}_n, \xi'_{n-1}\rangle \qquad (38)$$

in the space $\mathcal{V}^{\otimes}$, where k runs over the set $+, -, 3, \dots, n$, and coefficients $(k, (\mathbf{m}_n, \xi'_{n-1}) | (\mathbf{m}'_n, \xi_{n-1}); \epsilon)$ are Clebsch–Gordan coefficients (CGC's). Now we define these CGC's in an explicit form.

We put $(k, (\mathbf{m}_n, \xi'_{n-1}) | (\mathbf{m}'_n, \xi_{n-1}); \epsilon) = 0$ if one of the conditions

1) $\mathbf{m}'_n \notin \mathcal{S}(\mathbf{m}_n)$,
2) $\mathbf{m}_s \notin \mathcal{S}(\mathbf{m}'_s), s = n-1, \dots, k, \quad k \geq 3$,
3) $\mathbf{m}_s \notin \mathcal{S}(\mathbf{m}'_s), s = n-1, \dots, 3, \quad k = +, -$,
4) $\xi'_{k-1} \neq \xi_{k-1}, k = 3, 4, \dots, n$,
5) $m_{12} \neq m'_{12} + 1, k = +$,
6) $m_{12} \neq m'_{12} - 1, \quad k = -, \quad m'_{12} \geq \frac{3}{2}$,
6′) $m_{12} \neq m'_{12}, \quad k = -, \quad m'_{12} = \frac{1}{2}$.

is fulfilled. The nonzero CGC for $k = n$ are:

$$\begin{aligned}
&(2p{+}1, (\mathbf{m}_{2p+1}, \xi_{2p}) | (\mathbf{m}^{+j}_{2p+1}, \xi_{2p}); \epsilon) {=} \Big(\textstyle\prod_{r=1}^{p} [l_{j,2p+1} + l_{r,2p}][l_{j,2p+1} - l_{r,2p}]\Big)^{\frac{1}{2}}, \\
&(2p{+}1, (\mathbf{m}_{2p+1}, \xi_{2p}) | (\mathbf{m}_{2p+1}, \xi_{2p}); \epsilon) {=} \textstyle\prod_{r=1}^{p} [l_{r,2p}]_+, \\
&(2p{+}1, (\mathbf{m}_{2p+1}, \xi_{2p}) | (\mathbf{m}^{-j}_{2p+1}, \xi_{2p}); \epsilon) {=} \Big(\textstyle\prod_{r=1}^{p} [l_{j,2p+1} {+} l_{r,2p} {-} 1][l_{j,2p+1} {-} l_{r,2p} {-} 1]\Big)^{\frac{1}{2}},
\end{aligned} \qquad (39)$$

$$\begin{aligned}
&(2p, (\mathbf{m}_{2p}, \xi_{2p-1}) | (\mathbf{m}^{+j}_{2p}, \xi_{2p-1}); \epsilon) {=} \Big(\textstyle\prod_{r=1}^{p-1} [l_{j,2p} + l_{r,2p-1}][l_{j,2p} - l_{r,2p-1} + 1]\Big)^{\frac{1}{2}}, \\
&(2p, (\mathbf{m}_{2p}, \xi_{2p-1}) | (\mathbf{m}^{-j}_{2p}, \xi_{2p-1}); \epsilon) {=} \Big(\textstyle\prod_{r=1}^{p-1} [l_{j,2p} + l_{r,2p-1} - 1][l_{j,2p} - l_{r,2p-1}]\Big)^{\frac{1}{2}}, \\
&(2p, (\mathbf{m}_{2p}, \xi_{2p-1}) | (\mathbf{m}_{2p}, \xi_{2p-1}); \epsilon) {=} \textstyle\prod_{r=1}^{p-1} [l_{r,2p-1} - \frac{1}{2}], \qquad \text{if } m_{p,2p} = \frac{1}{2}.
\end{aligned} \qquad (40)$$

(They are defined up to normalization, that is, multiplication of these CGC's by some constants will not spoil the following results.)

All the other CGC's can be found from just presented by the following formula:

$$\begin{aligned}
(k, \xi_n | \xi'_n; \epsilon) = q^{k-n} &\frac{\langle \mathbf{m}_{n+1}, \xi_n | T_{\tilde{\epsilon}, \mathbf{m}_{n+1}}(I^-_{n+1,k}) | \mathbf{m}_{n+1}, \xi'_n \rangle}{\langle \mathbf{m}_{n+1}, \mathbf{m}_n, \xi_{n-1} | T_{\tilde{\epsilon}, \mathbf{m}_{n+1}}(I_{n+1,n}) | \mathbf{m}_{n+1}, \mathbf{m}'_n, \xi_{n-1} \rangle} \\
&\times (n, (\mathbf{m}_n, \xi_{n-1}) | (\mathbf{m}'_n, \xi_{n-1}); \epsilon),
\end{aligned} \qquad (41)$$

where the generators $I^-_{n+1,k}$ are defined in (2), $\tilde\epsilon=(\epsilon_2,\epsilon_3,\dots,\epsilon_n,+1)$. If $k=+$ or $k=-$ in the left-hand side of (41), one must put $k=2$ in right-hand side. The set $\mathbf{m}_{n+1}$ must be chosen to give non-zero denominator in right-hand side of (41). Note that if $(n,(\mathbf{m}_n,\xi_{n-1})|(\mathbf{m}'_n,\xi_{n-1}))\neq 0$, one can always do such a choice, moreover, the resulting CGC will not depend on this particular choice. In the case $n=3$ we reobtain the CGC's corresponding to the nonclassical type representations for the algebra $U'_q(\mathrm{so}_3)$ (see section 7).

Theorem 2. *The formulas for the action of the operators $T^{\otimes}(I_{k+1,k})$, $k=1,2,\dots,n-1$, on the vectors $|\mathbf{m}'_n,\xi_{n-1}\rangle^{\otimes}$ defined by (38) with CGC's defined by (39)–(41), coincide with the corresponding formulas (20)–(21) for the action of the operators $T_{\epsilon,\mathbf{m}'_n}(I_{k+1,k})$ on the GT basis vectors $|\mathbf{m}'_n,\xi_{n-1}\rangle$. We have the decomposition (34).*

9. The Wigner–Eckart theorem for the vector operators

To fix idea, we restrict ourselves to the case when vector operator acts on the space where direct sum of classical type representations of $U'_q(\mathrm{so}_n)$ is realized.

The formula (28) give us the transformation from the basis $\{v_k\otimes|\xi_n\rangle\}$ to the basis $\{|\xi'_n\rangle^{\otimes}\}$ in the space $\mathcal{V}^{\otimes}$. Because of (27), the transformation matrix is non-degenerate matrix with matrix elements being CGC's $(k,\xi_n|\xi'_n)$. Denote the matrix elements of inverse matrix by $(\xi'_n|k,\xi_n)$ (*inverse* CGC's). Let us find the expression for the vector $v_n\otimes|\mathbf{m}_n,\xi_{n-1}\rangle$ from (28) in terms of vectors $|\xi'_n\rangle^{\otimes}$. Since this vector transforms under the action of $T^{\otimes}(a)$, $a\in U'_q(\mathrm{so}_{n-1})$, as the vector $|\xi_{n-1}\rangle$ under the action of $T_{\mathbf{m}_{n-1}}(a)$ (see formula (5)), from Schur lemma we have

$$v_n\otimes|\mathbf{m}_n,\xi_{n-1}\rangle=\sum_{\mathbf{m}'_n}((\mathbf{m}'_n,\xi_{n-1})|n,(\mathbf{m}_n,\xi_{n-1}))\,|\mathbf{m}'_n,\xi_{n-1}\rangle^{\otimes},\tag{42}$$

where the coefficients $((\mathbf{m}'_n,\xi_{n-1})|n,(\mathbf{m}_n,\xi_{n-1}))$ depend only on $\mathbf{m}'_n$, $\mathbf{m}_n$, $\mathbf{m}_{n-1}$. From (28), it also follows that $\mathbf{m}'_n\in\mathcal{S}(\mathbf{m}_n)$. Although these coefficients are uniquely defined by (28)–(31), we shall need only their explicit dependence on $\mathbf{m}_{n-1}$.

Definition 1. *The set $\{V_k\}$, $k=1,2,\dots,n$, of operators on $\mathcal{V}$, where a representation T of $U'_q(\mathrm{so}_n)$ is realized, such that*

$$[V_{j-1},T(I_{j,j-1})]_q=V_j,\quad [T(I_{j,j-1}),V_j]_q=V_{j-1},\tag{43}$$

$$[T(I_{j,j-1}),V_k]=0,\qquad \textit{if } j\neq k \textit{ and } j-1\neq k,\tag{44}$$

where $[X,Y]_q=q^{1/2}XY-q^{-1/2}YX$, is called vector operator of the algebra $U'_q(\mathrm{so}_n)$.

It is easy to verify, that the action of operators $T(I_{j,j-1})$ on the vectors $V_k\,v_\alpha$ directly correspond to the action (3)–(5) of operators $T^{\otimes}(I_{j,j-1})$ on the vectors $v_k\otimes v_\alpha$.

Let T be a direct sum of irreducible classical type representations of $U'_q(\mathrm{so}_n)$ with arbitrary multiplicities. Choose Gel'fand–Tsetlin (GT) basis in $\mathcal{V}$. Let us consider an invariant subspace $\mathcal{V}_{\mathbf{m}_n,s}$ where subrepresentation equivalent to $T_{\mathbf{m}_n}$ is realized. The number s labels the number of such subspace if the corresponding multiplicity exceeds 1. Combine the vectors $V_k\,|(\mathbf{m}_n,\xi_{n-1});s\rangle$, where $\{|(\mathbf{m}_n,\xi_{n-1});s\rangle\}$ is GT basis of $\mathcal{V}_{\mathbf{m}_n,s}$, with CGC as in (28) for some fixed $\mathbf{m}'_n\in\mathcal{S}(\mathbf{m}_n)$. It is possible two variants. First,

all the vectors $|\mathbf{m}'_n, \xi_{n-1}\rangle^{\otimes}$ are zero. Second, on the space spanned by the vectors $|\mathbf{m}'_n, \xi_{n-1}\rangle^{\otimes}$, a representation of $U'_q(\mathrm{so}_n)$ equivalent to $T_{\mathbf{m}'_n}$ is realized. From Schur lemma, it follows that

$$|\mathbf{m}'_n, \xi_{n-1}\rangle^{\otimes} = \sum_{s'} (\mathbf{m}'_n, s' \| V \| \mathbf{m}_n, s)\, |\mathbf{m}'_n, \xi_{n-1}; s'\rangle, \tag{45}$$

where $(\mathbf{m}'_n, s' \| V \| \mathbf{m}_n, s)$ are some coefficients (*reduced matrix elements*) depending only on $\mathbf{m}'_n$, s', $\mathbf{m}_n$, s and vector operator $\{V_k\}$. Using the analogue of relation (42) for vector operator and (45) we have

$$\begin{aligned} V_n |\mathbf{m}_n, \xi_{n-1}; s\rangle = \sum_{\mathbf{m}'_n, s'} & ((\mathbf{m}'_n, \xi_{n-1}) | n, (\mathbf{m}_n, \xi_{n-1})) \\ & \times (\mathbf{m}'_n, s' \| V \| \mathbf{m}_n, s)\, |\mathbf{m}'_n, \xi_{n-1}; s'\rangle. \end{aligned} \tag{46}$$

As was claimed above, the coefficients $((\mathbf{m}'_n, \xi_{n-1}) | n, (\mathbf{m}_n, \xi_{n-1}))$ may depend on $\mathbf{m}_{n-1}$. Since this dependence is identical for all the possible vector operators in arbitrary spaces, we choose, for a moment, $\mathcal{V}$ to be the space $\mathcal{V}_{\mathbf{m}_{n+1}}$ of irreducible representation $T_{\mathbf{m}_{n+1}}$ of $U'_q(\mathrm{so}_{n+1})$ for some convenient $\mathbf{m}_{n+1}$, and $\{V_k\} \equiv \{T_{\mathbf{m}_{n+1}}(I^+_{n+1,k})\}$. Extracting the dependence on $\mathbf{m}_{n-1}$ from the matrix elements of $T_{\mathbf{m}_{n+1}}(I_{n+1,n})$ and comparing it with formulas (29)–(30), we obtain

$$((\mathbf{m}'_n, \xi_{n-1}) | n, (\mathbf{m}_n, \xi_{n-1})) = (n, (\mathbf{m}_n, \xi_{n-1}) | (\mathbf{m}'_n, \xi_{n-1})) \lambda_{\mathbf{m}'_n, \mathbf{m}_n},$$

where $\lambda_{\mathbf{m}'_n, \mathbf{m}_n}$ are some coefficients depending on $\mathbf{m}'_n$ and $\mathbf{m}_n$ only. Returning to the formula (46) and denoting $(\mathbf{m}'_n, s' \| V \| \mathbf{m}_n, s)' = (\mathbf{m}'_n, s' \| V \| \mathbf{m}_n, s) \times\, \lambda_{\mathbf{m}'_n, \mathbf{m}_n}$ we have

$$\begin{aligned} V_n |\mathbf{m}_n, \xi_{n-1}; s\rangle = \sum_{\mathbf{m}'_n, s'} & (n, (\mathbf{m}_n, \xi_{n-1}) | (\mathbf{m}'_n, \xi_{n-1})) \\ & \times (\mathbf{m}'_n, s' \| V \| \mathbf{m}_n, s)'\, |\mathbf{m}'_n, \xi_{n-1}; s'\rangle. \end{aligned} \tag{47}$$

Iterating the second formula in (43), we obtain the action formulas for $\{V_k\}$, $1 \le k < n$. Thus, we deduce the following q-analogue of Wigner–Eckart theorem.

Theorem 4. *If $\mathcal{V}$ is a Hilbert space and its Gel'fand–Tsetlin basis $\{|\mathbf{m}_n, \xi_{n-1}; s\rangle\}$ is orthonormal, we have, for the components of vector operator $\{V_k\}$ on $\mathcal{V}$, the decomposition*

$$\langle \mathbf{m}'_n, \xi'_{n-1}; s' | V_k | \mathbf{m}_n, \xi_{n-1}; s\rangle = ((\mathbf{m}'_n, \xi'_{n-1}) | k, (\mathbf{m}_n, \xi_{n-1}))' (\mathbf{m}'_n, s' \| V \| \mathbf{m}_n, s)',$$

where

$$((\mathbf{m}'_n, \xi'_{n-1}) | k, (\mathbf{m}_n, \xi_{n-1}))' = \frac{\langle \mathbf{m}_{n+1}, \xi'_n | T_{\mathbf{m}_{n+1}}(I^+_{n+1,k}) | \mathbf{m}_{n+1}, \xi_n\rangle}{\langle \mathbf{m}_{n+1}, \mathbf{m}'_n, \xi_{n-1} | T_{\mathbf{m}_{n+1}}(I_{n+1,n}) | \mathbf{m}_{n+1}, \mathbf{m}_n, \xi_{n-1}\rangle}$$

$$\times (n, (\mathbf{m}_n, \xi_{n-1}) | (\mathbf{m}'_n, \xi_{n-1})), \qquad 1 \le k < n$$

(see comments after analogous formula (31)).

Let T_ϵ be a direct sum of irreducible nonclassical type representations of $U'_q(\mathrm{so}_n)$ with arbitrary multiplicities and fixed ϵ on the Hilbert space $\mathcal{V}_\epsilon$. Choose Gel'fand–Tsetlin basis in $\mathcal{V}_\epsilon$. The the space $\mathcal{V}_\epsilon$ is direct sum of subspaces $\mathcal{V}_{\epsilon,\mathbf{m}_n,s}$ where subrepresentations equivalent to $T_{\epsilon,\mathbf{m}_n}$ are realized. The number s labels the number of such subspace if the corresponding multiplicity exceeds 1. Using argumentation analogous to the case of classical type representations, we derive the following q-analogue of Wigner–Eckart theorem for the case of nonclassical type representations.

Theorem 5. *If $\mathcal{V}_\epsilon$ is a Hilbert space and its Gel'fand–Tsetlin basis $\{|\mathbf{m}_n,\xi_{n-1};s\rangle\}$ is orthonormal, we have, for the components of vector operator $\{V_k\}$ on $\mathcal{V}_\epsilon$, the decomposition*

$$\langle \mathbf{m}'_n,\xi'_{n-1};s'|V_k|\mathbf{m}_n,\xi_{n-1};s\rangle = ((\mathbf{m}'_n,\xi'_{n-1})|k,(\mathbf{m}_n,\xi_{n-1});\epsilon)'$$
$$\times(\epsilon,\mathbf{m}'_n,s'\|V\|\epsilon,\mathbf{m}_n,s)',$$

where

$$((\mathbf{m}'_n,\xi'_{n-1})|k,(\mathbf{m}_n,\xi_{n-1});\epsilon)' =$$
$$\frac{\langle \mathbf{m}_{n+1},\xi'_n|T_{\tilde\epsilon,\mathbf{m}_{n+1}}(I^+_{n+1,k})|\mathbf{m}_{n+1},\xi_n\rangle}{\langle \mathbf{m}_{n+1},\mathbf{m}'_n,\xi_{n-1}|T_{\tilde\epsilon,\mathbf{m}_{n+1}}(I_{n+1,n})|\mathbf{m}_{n+1},\mathbf{m}_n,\xi_{n-1}\rangle}$$
$$\times(n,(\mathbf{m}_n,\xi_{n-1})|(\mathbf{m}'_n,\xi_{n-1});\epsilon),\qquad 1\le k<n$$

(see comments after analogous formula (41)).

The coefficients $(\epsilon,\mathbf{m}'_n,s'\|V\|\epsilon,\mathbf{m}_n,s)'$ are *reduced matrix elements* for the vector operator $\{V_k\}$.

If the representation T is a direct sum of classical type representations and nonclassical type representations with different ϵ, it is easy to find the matrix elements for the vector operators. It is sufficient to take into account the fact that vector operator 'acting' on classical type representation can not give nonclassical type representation, and 'acting' on nonclassical type representation with some set ϵ can not give nonclassical type representation with other set ϵ'. Thus, corresponding matrix elements are zero. The non-zero matrix elements are described by Theorem 4 and Theorem 5.

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